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OPTIMUM-RATE PROCESSES WITH BOUNDED PHASE COORDINATES

-USSR-

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(Presented by Academician L.S. Pontryagin 28 December 1958)

1. The Maximum Principle. References 1-6 contain the results of studies on the theory of optimal processes being conducted within the framework of L.S. Pontryagin's seminar on oscillation and automatic control theory. The final stage in these studies consisted in the proof of the general maximum principle (5,6) giving the necessary condition satisfied by any solution of the following optimal problem.

Let the vector function $f(x,u) = (f^1(x,u),\ldots,f^n(x,u))$ in variables x and u be determined and continuous over any direct product $(x,u) \in X^n \cdot \Omega$, $x \in X^n$, $u \in \Omega$, where X^n is the n-dimensional phase space of the problem, Ω is an arbitrary Hausdorff topological space of possible values for the control parameter u; in addition to this, it is assumed that that functions $f^i(x,u)$, $i = 1,\ldots,n$, are continuously differentiable at all points (x,u) along all coordinates of vector $x = (x^1,\ldots,x^n)$.

The equation of motion of the phase point x has the form

$$\dot{x} = f(x,u). \qquad (1)$$

Two points ξ_1 and ξ_2 are taken in X^n. The task is one of

selecting from a class of permissible control functions (for example, the class of measurable limited or piecewise continuous controls) a function $u(t)$, $T_1 \leqslant t \leqslant T_2$ (T_1 and T_2 arbitrary) for which the corresponding locus $x(t)$ of equation (1) connects points ξ_1, ξ_2 and the integral $\int_{T_2}^{T_1} L(x(t),u(t))\,dt$ assumes a minimum value.

At $L(x,u) \equiv 1$, the problem stated above becomes an optimum-rate problem (1,4).

In this case, the Maximum Principle is formulated in the following manner.

If $x(t)$, $t_1 \leqslant t \leqslant t_2$ is the rate-optimal locus of equation (1), and $u(t)$, $t_1 \leqslant t \leqslant t_2$ is the correspondting optimal control, then there will be a continuous, non-vanishing covariant vector function $\psi(t) = (\psi_1(t),\ldots,\psi(t))$, $t_1 \leqslant t \leqslant t_2$ such that the following Hamiltonian system will hold on the interval $t_1 \leqslant t \leqslant t_2$:

$$\dot{x}^i = \frac{\partial H}{\partial \psi_i}, \qquad \dot{\psi}_i = \frac{\partial H}{\partial x^i},$$

$$H(x(t),\psi(t), u(t)) = M(x(t), \psi(t)) = \mathrm{const} \geqslant 0, \quad (2)$$

where the Hamiltonian $H(x, \psi, u) = \psi \cdot f(x,u) = \psi_\alpha f^\alpha(x,u)$, $M(x, \psi) = \sup_{u \in \Omega} H(x, \psi, u)$.

The space of possible values Ω for the control parameter can, in one instance, be the closed domain of an r-dimensional linear space. The set of possible values for the phase point x, moreover, must correspond with the entire X^n space; in the contrary case, the Maximum Principle ceases to hold true. However, the case wherein the set of possible phase point values comprises a closed region X^n with piecewise smooth bounds has extremely important applications.

The present brief paper contains a formulation of the results obtained by the author on optimum-rate processes with bounded phase coordinates as part of his work in L.S. Pontryagin's seminar. For the sake of simplicity, the case examined here is one for which the region of possible phase point values has smooth bounds. The case of the piecewise-

continuous bounds is treated in an analogous manner. The transition from the particular case examined in the present to the general integral minimization problem is similar to the transition in the problem treated in (5,6).

2. The Statement of the Problem. Let Ω be a bounded region in the r-dimensional linear space $E^r \ni u = (u^1,\ldots,u^r)$, specified by a system of inequalities $q_i(u) \leq 0$, $i = 1,\ldots,m$.

The class of permissible controls consists of all piecewise-continuous, piecewise-smooth vector functions $u(t) = (u^1(t),\ldots,u^r(t))$ with first-order discontinuities determined on an arbitrary interval $t_1 \leq t \leq t_2$ and with values in Ω at any instant in time. In the phase space X^n of the formulated problem, let there be given a closed region G with smooth bounds determined by the inequality $g(x^1,\ldots,x^n) = g(x) \leq 0$, where the function $g(x)$ has continuous partial second derivatives in the region of the boundary $g(x) = 0$ and the vector grad $g(x) = (\partial g/\partial x^1,\ldots,\partial g/\partial x^n) = (g_1(x),\ldots,g_n(x))$ does not go to zero.

The function of motion of phase point $x = (x^1,\ldots,x^n) \in X^n$ is given in terms of the following normal system of differential equations:

$$\dot{x}^i = f^i(x,u), \qquad (3)$$

where the vector function $f(x,u) = (f^1(x,u),\ldots,f^n(x,u))$ is determined on the direct product $G^* \cdot \Omega^*$, where G^* and Ω^* are open sets in the spaces X^n and E^r containing G and Ω respectively and continuously differentiable therein over all x and u vector coordinates.

The Formulation of the Problem. In phase space X^n are given points ξ_1 and ξ_2 lying in the closed region G; it is necessary to select a possible control such that the phase point moving along the locus of system (3) lying entirely within the closed region G will move from position ξ_1 to position ξ_2 in the minimum time.

Let us call this control the optimal control and the corresponding locus, the optimal locus.

3. Optimal loci on the boundary of region G. Let us introduce the designations $p(x,u) = \text{grad } g(x) \cdot f(x,u) = g_\alpha(x) f^\alpha(x,u)$, and grad $p(x,u) = (\partial p/\partial u^1,\ldots,\partial p/\partial u^r)$. In order for the locus $x(t)$ of the system (3), corresponding to the control $u(t)$, $t_1 \leq t \leq t_2$, to lie on the boundary $g(x) = 0$ of the region G it is necessary and sufficient that

$p(x(t),u(t)) \equiv 0,\ t_0 \leq t \leq t_2,\ g(x(t_1)) = 0.$

Let us call point x on the boundary $g(x) = 0$ a regular point relative to point $u \in \Omega$ which satisfies the conditions

$$q_{i_1}(u) = \dots = q_{i_s}(u) = 0,\ q_j(u) \quad 0,\ j \neq i_1,\dots,i_s \quad (4)$$

if $p(x,u) = 0$ and vectors grad $p(x,u)$, grad $q_{i_1}(u),\dots,$ grad $q_{i_s}(u)$ are independent. Let us denote by means of $\omega(x)$ the set of those $u \in \Omega$ relative to which the point x is regular. The locus $x(t)$, $t_1 \leq t \leq t_2$ of system (3), corresponding to control $u(t)$ and lying entirely on the boundary $g(x) = 0$ is to be called regular if $u(t) \in \omega(x(t))$, $t_1 \leq t \leq t_2$. By means of $\psi = (\psi_1,\dots,\psi_n)$ let us denote the covariant vector of the space X^n. Let x lie on the boundary $g(x) = 0$. The upper bound of the function $H(x,\psi,u) = \psi \cdot f(x,u) = \psi_\lambda \cdot f^\alpha(x,u)$ with fixed x, ψ, u and variable $\omega(x)$ will be denoted by $m(x,\psi)$: $m(x,\psi) = \sup_{u \in \omega(x)} H(x,\psi,u)$. (We shall be interested only in those points x on the boundary $g(x) = 0$ for which $\omega(x)$ is not empty).

If x is a regular point on the boundary $g(x) = 0$ relative to u, where u satisfies the conditions (4) and $H(x,\psi,u) = m(x,\psi)$, then according to the Lagrange multiplier rule,

$$\text{grad } H(x,\psi,u) = (\partial H/\partial u^1,\dots,\partial H/\partial u^r) = \lambda \text{grad } p(x,u) + \sum_{k=1}^{s} \mu_k \text{grad } q_{i_k}(u). \quad (5)$$

The regular locus $x(t)$, $t_1 \leq t \leq t_2$ of system (3), corresponding to the permissible control $u(t)$, $t_1 \leq t \leq t_2$ and lying entirely on the boundary $g(x) = 0$ of region G, let us designate by the term <u>extremal locus</u>; $u(t)$ will then be called the corresponding <u>extremal control</u> if there exists a non-vanishing continuous piecewise smooth vector function $\psi(t) = (\psi_1(t),\dots,\psi_n(t))$, $t_1 \leq t \leq t_2$, such that the following system of equations is satisfied on the interval $t_1 \leq t \leq t_2$:

$$p(x,u) = 0,\quad \dot{x}^i = -\frac{\partial H}{\partial \psi_i},\ \dot{\psi}_i = -\frac{\partial H}{\partial x^i} + \lambda(t)\frac{\partial p}{\partial x^i}$$
$$H(x,\psi,u) = m(x,\psi) \geq 0 \quad (6)$$

where vector $\psi(t_1)$ is not collinear with the vector grad $g(x(t_1))$ and the piecewise smooth function $\lambda(t)$ is given at each instant of time by formula (5). In addition to this, extremality also requires that for each t in the interval $t_1 \leq t \leq t_2$, the vector $d\lambda/dt$ grad $g(x(t))$ be directed into the interior of region G, or else go to zero.

It is not difficult to prove that along any extremal locus $H(x(t), \psi(t), u(t)) = m(x(t), \psi(t)) = \text{const} \geq 0$.

Theorem I. Any regular optimal locus of system (3) lying entirely on the boundary of region G and the corresponding optimal control are extremal.

4. Jump conditions. Let $x(t)$, $t_1 \leq t \leq t_2$ be an optimal locus lying in (closed) region G. Let it be regular in every segment lying on the boundary of the region. The point $x(\tau)$ of the locus lying on the boundary of the region will be called a junction point provided that $t_1 \leq \tau \leq t_2$ and there is an $\varepsilon > 0$ such that at least one of the locus $x(t)$ segments lies in the open kernel of region G for $\tau - \varepsilon < t < \tau$ or $\tau < t < \tau + \varepsilon$. For the sake of concreteness, let us assume that a portion of the locus belongs to the open kernel of the region for $\tau - \varepsilon < t < \tau$. Let us call τ the junction time. Let $x(\tau)$ be the only junction point of the optimal locus $x(t)$, $t_1 \leq t \leq t_2$; $u(t)$ is the corresponding optimal control. For $t_1 < t < \tau$, a segment of the locus lies in the open kernel of region G. In the case where $\tau \leq t < t_2$, the segment either lies entirely on the boundary of the region or the segment $\tau < t < t_2$ is also included in the open kernel of the region and the point $x(t)$ is the only point of the locus lying on the boundary of the region (with the possible exception of the end points).

Consequently, the segment $x(t)$, $t_1 \leq t \leq \tau$, satisfies the Maximum Principle (paragraph #1). The vector function $\psi(t)$ which corresponds to this segment is continuous on the interval $t_1 \leq t \leq \tau$, and the function $H(x(t), \psi(t), u(t)) = \text{const} = c \geq 0$ for $t_1 \leq t \leq \tau$. The segment of $x(t)$ for $\tau \leq t \leq t_2$ likewise satisfies either system (6) or (2); the vector function $\bar{\psi}(t)$ corresponding to this segment is also continuous over the segment $\tau \leq t \leq t_2$, and the function $H(x(t), \bar{\psi}(t), u(t)) = \text{const} = \bar{c} \geq 0$ for $\tau \leq t \leq t_2$.

We will say that at an isolated junction point $x(\tau)$ of the regular optimal trajectory $x(t)$, $t_1 \leq t \leq t_2$, the jump condition is satisfied, provided that there exists a segment of $x(t)$, $t_3 \leq t \leq t_4$ such that the interval $t_3 < t < t_4$ represents

the maximum interval of the segment $t_1 \leq t \leq t_2$ which contains the single junction instant $\mathcal{T}$, and the vector functions $t_3 \leq t \leq \mathcal{T}$ defined above; $\overline{\psi}(t), \mathcal{T} \leq t \leq t_4$ can be chosen in a way such that one of the following pair of relationships is satisfied:

$$\overline{\psi}(\mathcal{T}) - \psi(\mathcal{T}) = \mu \operatorname{grad} g(x(\mathcal{T})), \quad c = \overline{c}; \qquad (7)$$

$$\psi(\mathcal{T}) = \mu \operatorname{grad} g(x(\mathcal{T})), \quad c = 0, \qquad (8)$$

where μ is a real number.

Theorem II. Let the regular optimal locus lying in the closed region G contain a finite number of junction points. Then the jump condition will be satisfied at every junction point.

5. General rule for the determination of regular optimal loci. Combining the Maximum Principle (paragraph #1) with theorems I and II, we arrive at the following general rule for determining regular optimal loci.

Let the optimal locus $x(t)$ lie entirely within the closed region G, containing a finite number of junction points, and with every segment lying on the boundary of the region regular. Then any segment of the locus lying entirely within the open kernel of region G (with the possible exception of the end points) satisfies the maximum condition (paragraph #1); any of its segments lying on the boundary of region G is an extremum (in the sense of paragraph 3); the jump condition is satisfied at each junction point.

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